SOLUTION OF THE LINEAR ONE-DIMENSIONAL INVERSE HEAT-CONDUCTION PROBLEM ON THE BASIS OF A HYPERBOLIC EQUATION

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The linear problem of determining the temperature in an infinite one-dimensional plate with a stationary heat sensor and a stationary boundary is solved. The allowable approximation time steps in the calculations can be diminished by using a hyperbolic heat-conduction equation with a suitably chosen hyperbolicity parameter.

It has been shown [1] that using a hyperbolic heat-conduction equation to solve the linear inverse heat-conduction problem (IHCP) yields the well-posed problem of solving a Volterra integral equation of the second kind. We now extend this approach to the case of an infinite plate.

We consider the heat-conduction process in an infinite one-dimensional plate $x \in [0, b]$, as described by the hyperbolic equation with constant coefficients [2, 3]

$$\beta^2 \frac{\partial^2 u}{\partial t^2} + \frac{1}{a} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

subject to the initial conditions

$$u(0, x) = 0; \quad \frac{\partial u(0, x)}{\partial x} = 0. \tag{1a}$$

One of the following three boundary conditions is prescribed at the boundary of the domain (at x = 0):

$$u(t, 0) = u_0(t); \; q(t, 0) = q_0(t); \; q(t, 0) = \alpha [u_0(t) - u(t, 0)]. \tag{1b}$$

At the point $x = x_1$ there is a temperature sensor, so that the following measurement condition is given there:

$$u(t, x_1) = u_1(t). \tag{1c}$$

In expressions (1b) and (1c) $u_0(t)$, $q_0(t)$, and $u_1(t)$ can be arbitrary functions of the time. Equation (1) in conjuction with conditions (1a)-(1c) comprises a direct boundary-value problem for the domain $0 \leqslant x \leqslant x_1$ of the plate and therefore uniquely determines the temperature field in that domain. We assume that the temperature field for $0 \leqslant x \leqslant x_1$ is known (it can be determined by any numerical method [4]), and so the derivative $\partial u(t, x_1)/\partial x$ and the heat flux $q(t, x_1)$ at the point $x = x_1$ are known. We note that $q(t, x)$ is related to the temperature $u(t, x)$ inside the plate as follows [2, 3]:

$$\eta \frac{\partial q}{\partial t} + q = -\lambda \frac{\partial u}{\partial x}; \quad \eta = a\beta^2. \tag{2}$$

In the domain $x_1 \leqslant x \leqslant b$ the temperature $u(t, x)$ is obtained as the solution of the problem of continuing the temperature field to the boundary x = b according to the known temperature $u(t, x_1)$ and heat flux $q(t, x_1)$ at the point $x = x_1$. In place of q it is more practical to use the derivative $q_1^*(t) = \partial u(t, x_1)/\partial x$. Equation (1) in conjunction with the initial conditions (1a) and the conditions

$$u(t, x_1) = u_1(t); \quad \frac{\partial u(t, x_1)}{\partial x} = q_1^*(t) \tag{3}$$

comprises the inverse heat-conduction problem for the domain $x_1 \leqslant x \leqslant b$.

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We apply the Laplace transform to Eq. (1) and conditions (1a) and (3). Denoting the Laplace transf orms of the functions u(t, x), $u_1(t)$, $q_1^*(t)$ by the corresponding upper-case letters U(p, x), $U_1(p)$, $Q_1^*(p)$, we obtain an expression for U(p, x) in the domain $x_1 \leqslant x \leqslant b$:

$$U(p,\ x) = \sum_{k=1}^{2} A_k(p) \exp[(-1)^k \beta (x - x_1) \sqrt{p^2 + p/\eta}], \tag{4}$$

where

$$A_k(p) = U_1(p)/2 + \frac{(-1)^k Q_1^*(p)}{2\beta \sqrt{p^2 + p/\eta}};\ k = 1,\ 2. \tag{4a}$$

Multiplying both sides of Eq. (4) by $\exp[-\beta(x - x_1)\sqrt{p^2 + p/\eta}]$, we arrive at the inverse transforms therein, invoking the convolution theorem and the known Laplace transforms [1, 5]. The left-hand side of Eq. (4) is transformed into the operator of the Volterra integral equation for the function u(t, x), coinciding with the right-hand side of expression (7a) in [1] under replacement of the coordinate x by $x - x_1$. The right-hand side of (4) is transformed into the free term of a two-term integral equation. Introducing the dimensionless time $Fo = a t/l^2$, the quantity $\gamma = a\beta/l$, which is the dimensionless receiprocal of the heat propagation rate, the coordinate $X = (x - x_1)/l$, and the quantity $Fo^* = Fo - \gamma X$, we write the resulting integral equation in the form

$$u(\mathrm{Fo}^*,\ X) \exp(-X/2\gamma) + \int_0^{\mathrm{Fo}^*} u(\mathrm{Fo}',\ X) K(\mathrm{Fo}^* - \mathrm{Fo}' + \gamma X,\ X,\ \gamma)\, d\,\mathrm{Fo}' =$$
$$= \varphi_1(\mathrm{Fo}^* + \gamma X,\ X) + \mathrm{E}(\mathrm{Fo}^* - \gamma X)\, \varphi_2(\mathrm{Fo}^* + \gamma X,\ X). \tag{5}$$

Here we have the notation

$$\varphi_1(\mathrm{Fo},\ X) = \frac{l}{2\pi\gamma} \int_0^{\mathrm{Fo}} q_1^*(\mathrm{Fo} - \mathrm{Fo}')\, d\,\mathrm{Fo}' \int_0^{\mathrm{Fo}'} \frac{\exp(-\mathrm{Fo}^*/2\gamma^2)}{\sqrt{\mathrm{Fo}^*}\sqrt{\mathrm{Fo}' - \mathrm{Fo}^*}}\, d\,\mathrm{Fo}^* + \frac{1}{2} u_1(\mathrm{Fo});$$

$$\varphi_2(\mathrm{Fo},\ X) = \frac{1}{2} u_1(\mathrm{Fo} - 2\gamma X) \exp(-X/\gamma) + \frac{1}{2} \int_0^{\mathrm{Fo}-2\gamma X} u_1(\mathrm{Fo}') K(\mathrm{Fo} - \mathrm{Fo}',\ 2X,\ \gamma)\, d\,\mathrm{Fo}' +$$
$$+ \frac{l}{2\gamma} \int_0^{\mathrm{Fo}-2\gamma X} q_1^*(\mathrm{Fo}') K_1(\mathrm{Fo} - \mathrm{Fo}',\ 2X,\ \gamma)\, d\,\mathrm{Fo}'; \tag{5a}$$

$$K(\mathrm{Fo},\ X,\ \gamma) = \frac{X}{2\gamma} \frac{I_1\left(\frac{1}{2\gamma^2}\sqrt{\mathrm{Fo}^2 - \gamma^2 X^2}\right)}{\sqrt{\mathrm{Fo}^2 - \gamma^2 X^2}} \exp(-\mathrm{Fo}/2\gamma^2);$$

$$K_1(\mathrm{Fo},\ X,\ \gamma) = I_0\left(\frac{1}{2\gamma^2}\sqrt{\mathrm{Fo}^2 - \gamma^2 X^2}\right) \exp(-\mathrm{Fo}/2\gamma^2),$$

where I_0 and I_1 are Bessel functions of the first kind of an imaginary argument and E(Fo) is the unit step function [5].

The absolute error Δu of determination of the temperature u(Fo, X) due to error of the right-hand side of Eq. (5) can be estimated in the usual way [6]:

$$|\Delta u| \leqslant \delta \exp[X/2\gamma + K_0\, \mathrm{Fo}_{\max} \exp(X/2\gamma)], \tag{6}$$

where

$$K_0 \geqslant \max_{\mathrm{Fo}\in[0,\mathrm{Fo}_{\max}]} |K(\mathrm{Fo},\ X,\ \gamma)| \tag{6a}$$

and δ is the maximum absolute error in the interval $\mathrm{Fo}\in[0,\ \mathrm{Fo}_{\max}]$. Relation (6) implies that the integral equation (5) is stable for $\gamma > 0$. Consequently, problem (1), (1a), (3) for the continuation of the temperature field into the half-plane $x > x_1$ is formally well-posed, because it meets all three requirements for well-posedness [7].

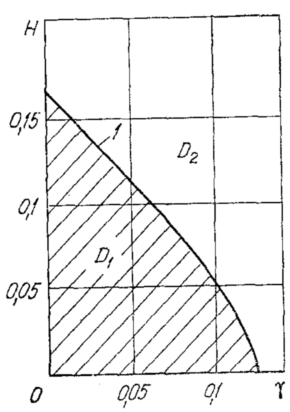

Fig. 1

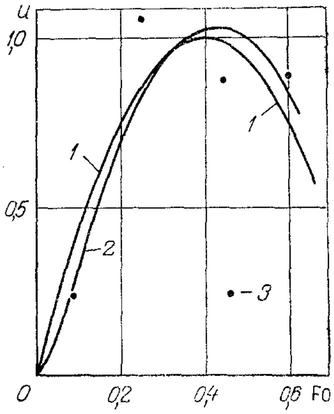

Fig. 2

Fig. 1. Partition of the plane HOγ into domains of stable and unstable (hatched) results. 1) Curve $H_{cr}(\gamma)$ plotted according to the criterion of Alifanov [8].

Fig. 2. Determination of the temperature at the boundary of a symmetrically heated infinite plate with heat source at $x_1 = b/2$. 1) Exact function u(Fo, 1); 2) temperature u(Fo, 1) determined according to expressions (7)-(9); 3) points calculated on the basis of the parabolic heat-conduction equation for H = 0.17.

To solve the integral equation (5) we can use the method of approximation of the unknown temperature u(Fo, X) by step functions $u_k = \{u(kH, X) + u[(k-1)H, X]\}/2$ in subintervals $B_k \in [(k-1)H, kH]$. In this case Eq. (5) is reduced to a system of linear algebraic equations with a triangular matrix, whose diagonal elements are equal to [1]

$$\sum_{l=1}^{k} d_{k-l}u_l = \varphi_{1k} + \varphi_{2k};\ k = 1,\ 2,\ \ldots,\ N;$$

$$d_k^* = \frac{1}{2}\int_0^H K(kH + \gamma X - \xi,\ X,\ \gamma)\,d\xi;\ k = 1,\ 2,\ \ldots,\ N; \tag{7}$$

$$d_0 = d_1^* + \exp(-X/2\gamma);\ d_{k-l} = d_{k-l}^* + d_{k-l+1}^*;\ k - l \geqslant 1,$$

where φ_{1k} and φ_{2k} are the arithmetic means of the corresponding functions φ_1 and φ_2 in the subinterval B_k. To calculate the quantities u_k it is convenient to use the recursion formula

$$u_k = \left[\varphi_{1k} + \varphi_{2k} - \sum_{l=1}^{k-1} d_{k-l}u_l\right] / d_0. \tag{8}$$

Thus, using the integral form of the hyperbolic heat-conduction equation and then reducing it to algebraic form, we transform the linear IHCT to a formally well-posed problem. Inasmuch as the relaxation times in solids are small, the values of the dimensionless reciprocal velocity γ usually fall within the limits $\gamma = 10^{-3}$ to 10^{-5}. For such small values of γ the algebraic system (7) is as ill-conditioned as for the parabolic heat-conduction equation. As a result, only for a large approximation time step H = 1/6 is it possible to obtain a stable solution of the IHCP for smooth input data. Consequently, allowance for a finite heat propagation rate corresponding to values of $\gamma = 10^{-3}$-10^{-5} scarcely improves the stability of the solution of the IHCP over the parabolic heat-conduction equation.

The condition of the algebraic system (7) improves with increasing value of γ, so that the integral equation (5) and the system (7) can be regarded as mathematical machinery for obtaining an approximate solution of the IHCP, given a suitable choice of the hyperbolicity parameter γ. While preserving the simplicity of the direct integral method of solution of the IHCP, this approach enables us to decrease the approximation time step, a feature that is important for the interpretation of dynamic thermal processes. A similar approach to the application of a hyperbolic heat-conduction equation, based on difference schemes, is used in [8], where the feasibility of decreasing the allowable computation steps with the use of smooth input data is demonstrated. A technique is proposed in [8] for determining the critical step H_{cr} for which the algebraic system approximating the Volterra integral equation of the first kind yields a stable solution. The function $H_{cr}(\gamma)$ is plotted in Fig. 1 according to this criterion for the integral equation (5). It separates the plane $HO\gamma$ into a domain D_2 of stable solution and a domain D_1 of unstable solution. Numerical experiments (see also [1]) have shown that for smooth input data, points of the plane $HO\gamma$ in the interior of D_1 can be used for solving the IHCP. This fact is a consequence of the additional stabilizing effect of the free term $u\exp(-X/2\gamma)$ in the integral equation (5). As γ tends to zero the error of approximation of the original parabolic heat-conduction equation by the integral equation (5) decreases, while the condition number of the algebraic system (7) increases. Consequently, a value γ_{opt} of the hyperbolicity parameter exists for which the error of determination of the temperature is a minimum. The optimal hyperbolicity parameter γ_{opt} corresponding to a good approximate solution of the IHCP can be chosen by a quasioptimization procedure. Having chosen the necessary approximation step H, from the curve $H_{cr}(\gamma)$ we find the corresponding value γ_0. Specifying the sequence of numbers $\gamma_l = \gamma_0\beta^l$ (where $0 < \beta < 1$ and $l = 0, 1, \ldots$), for each l we carry out calculations according to expressions (7) and (8) and choose the optimal value γ_{opt} of the hyperbolicity parameter from the condition of minimization of the maximum deviation:

$$\min_{l} \max_{k\in[1,N]} |u_k(\gamma_l) - u_k(\gamma_{l-1})|. \tag{9}$$

Numerical calculations indicate sufficient reproducibility of the temperature at the boundary of the plate with a small computing step for smooth input data. The results of one such calculation are shown in Fig. 2. The approximation step was taken equal to H = 0.04, and $\beta = 0.85$. The optimal value of the hyperbolicity parameter was $\gamma_{opt} = 0.05$. It is important to note that if the input data are distorted by errors, then the initial temperature is poorly reproduced. In this event it is required to take larger values of γ in order to obtain a stable solution ($\gamma \geqslant 0.125$), causing a departure from the true temperature. Such input data must therefore be approximated by a smooth function according to the technique described in [9]. We add in conclusion that allowance for the finite mass-transfer rate of material in inverse mass-transfer problems by means of a hyperbolic equation [10] should have a more effective regularizing action in comparison with heat-conduction problems.

NOTATION

λ, thermal conductivity; a, thermal diffusivity; β, reciprocal of the heat propagation rate in the body; p, Laplace variable; b, width of the plate; $l = b - x_1$, distance from the right boundary of the plate to the measurement point; $u_0(t)$, temperature of the medium at the left boundary of the plate; u(t, x), temperature in the interior of the plate; q(t, x), heat flux in the interior of the plate; $u_1(t)$, temperature measured in the cross section $x = x_1$; $q_0(t)$, heat flux applied to the left boundary of the plate; u(Fo, 1), unknown temperature function at the right boundary of the plate.

LITERATURE CITED

1. I. A. Novikov, "Solution of linear inverse heat-conduction boundary-value problems," Inzh.-Fiz. Zh., 35, No. 4, 529-535 (1978).
2. A. V. Lykov, Theory of Heat Conduction [in Russian], Vysshaya Shkola, Moscow (1967).
3. A. G. Temkin, Inverse Heat-Conduction Methods [in Russian], Énergiya, Moscow (1973).
4. V. I. Krylov, V. V. Bobkov, and P. I. Monastyrskii, Computational Methods [in Russian], Vol. 2, Nauka, Moscow (1977).
5. G. Doetsch, Handbook for the Practical Application of the Laplace and z-Transforms [Russian translation], Nauka, Moscow (1971).
6. V. I. Smirnov, Course in Higher Mathematics [in Russian], Vol. 4, GITTL, Moscow (1953).
7. O. M. Alifanov, "The inverse heat-conduction problem," Inzh.-Fiz. Zh., 25, No. 3, 530-537 (1973).
8. O. M. Alifanov, Identification of Aircraft Heat-Transfer Processes [in Russian], Mashinostroenie, Moscow (1979).

9. O. M. Alifanov, E. A. Artyukhin, and B. M. Pankratov, "Solution of the nonlinear inverse problem for the generalized heat-conduction equation in a domain with moving boundaries," Inzh.-Fiz. Zh., 29, No. 1, 151-158 (1975).
10. I. M. Taganov, Modeling of Mass and Energy Transport Processes [in Russian], Khimiya, Leningrad (1979).

PROPERTIES OF THE MOTION OF LIQUIDS IN POROUS MATERIALS (REVIEW)

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The possibility of precise and predictable control of the flow rate of a liquid in porous materials is one of the most important conditions for reliable operation of various heat- and mass-transfer systems based on the use of fluid flow in these materials. However, experiments on filtration of a fluid in porous materials have shown that an undesirable nonuniform and nonreproducible decrease in time in the flow is observed. In the literature, this phenomenon is known as the "filtration effect" [1] or, in analogy with the flow of a liquid through capillaries, "obliteration" [2] and is observed during motion of different liquids in porous materials [1-14]. A typical example of the decrease in the liquid flow through a porous specimen with a constant pressure differential is presented in Fig. 1.

It is of great interest to generalize and analyze the available experimental data in order to clarify the true reasons for the phenomenon indicated, about which different opinions still exist in the literature.

This problem is discussed most completely in [2, 3]. The following are the main reasons mentioned for the decrease in the liquid flow rate with flow in porous materials:

1) The physical properties (in particular, viscosity) of the liquid in a thin layer near a continuous surface differ from the physical properties in the bulk.

2) Molecular layers that gradually decrease the clear pore openings in the porous structure are adsorbed on the surface of the solid body.

When a liquid makes contact with a solid surface, adsorption films whose properties differ from the properties of the liquid in the bulk are formed. If the liquid is a nonpolar dielectric, whose molecules have a zero dipole moment, then the interaction between the molecules of the material and the liquid stems from molecular force fields. Such liquids include benzene, kerosene, etc. If the liquid is a polar dielectric, then the electric field of the material causes definite orientation of the molecules of the liquid and formation of a polymolecular adsorption layer. Such liquids include water, acetone, etc. If the liquid contains free ions, then due to their adsorption an electrically charged double-layer is formed.

Regardless of the reasons for the formation, changes are observed in the structure of the fluid in surface layers (ordered molecular layers), and therefore, changes in the structurally sensitive physical properties (in particular, viscosity and thermal conductivity). It follows from here that the first of the reasons mentioned above for obliteration is a result of the formation of adsorbed layers.

In order to verify the hypothesis of the considerable influence of an adsorbed layer on the decrease of the liquid flow rate in porous materials, it is necessary to have information concerning the thickness of this layer. Its thickness depends on the thermophysical and thermodynamic properties of the liquid and of the solid body, temperature, and structure of the porous material. By applying shear stresses (external flow), it is possible to decrease the thickness of the adsorbed layers due to detachment of the outer, weakly bound molecules. A gradual weakening of the boundary layers of the fluid with increasing tempera-

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